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Bounds on the momentum transport by turbulent shear flow in rotating systems

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Bounds on the momentum transport by laminar or turbulent shear flows between two parallel plates in constant relative motion in a rotating system are derived. The axis of rotation is parallel to the plates. The dimensionless component of the rotation vector perpendicular to the relative motion of the plate is denoted by the Coriolis number τ. Through the consideration of separate energy balances for the poloidal and the toroidal components of the fluid velocity field a variational problem is formulated in which τ enters as a parameter. Bounds that are derived under the hypothesis that the extremalizing vector fields are independent of the streamwise coordinate suggest that no state of turbulent motion can exist for $-2\sqrt{1708} \equiv -Re_E \leqslant Re \leqslant 1708/\tau + \tau$ with $\tau \gtrsim \sqrt{1708}$.

1. Introduction

Most bounds on turbulent transport in systems of incompressible fluids are derived independently of the property of a rotation of the system. This is because usually only the energy balance for the turbulent velocity field enters the derivation of the bounds. Because the Coriolis force does not do any work the parameter of rotation drops out of the energy balance. Experimentally realized turbulent flows exhibit considerable variations as a function of the rotation parameter, however, and improved bounds incorporating the influence of rotation are thus highly desirable. A convenient way in which the Coriolis force can be incorporated into an upper-bound analysis is the consideration of energy balances for separate parts of the fluctuating velocity field. In their derivation of a bound on the convective heat transport in a fluid layer heated from below, cooled from above and rotating about a vertical axis Vitanov & Busse (2001) used separate balances for the poloidal and toroidal components of the velocity field. In this case the variational problem for the determination of the upper bound had to be solved numerically, however, such that an extrapolation to the asymptotic regime of high Rayleigh numbers and high Coriolis numbers has been possible only with restrictions.

The separate use of poloidal and toroidal power constraints has long been a desideratum of the upper-bound theory (Busse 1978) and Kerswell & Soward (1996) have included both constraints in their derivation of an upper bound for the momentum transport in shear flows. In their case only a non-rotating system has been considered and the authors conclude that the bound could not be improved significantly through the use of separate energy balances for poloidal and toroidal components of the velocity field.

In this paper a bound is derived in which the rotation parameter plays an essential role. Through the consideration of separate energy balances for poloidal and toroidal

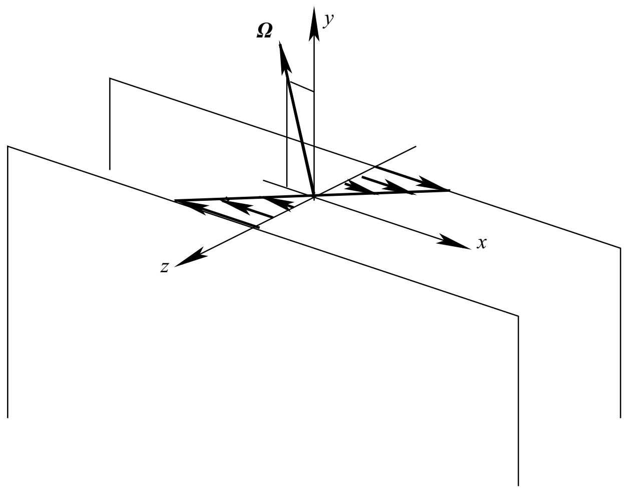

FIGURE 1. Sketch of the geometrical configuration of the problem.

components of the velocity field the upper bound becomes strongly dependent on the Coriolis number. This effect is strengthened by the assumption that the extremalizing vector fields do not depend on the streamwise coordinate of the problem. While this hypothesis is also made in the case of bounds that depend on a single energy balance and thus become independent of the Coriolis number, it is contradicted by experimental evidence for very low rotation rates. No such contradiction is known or expected for sufficiently high values of the rotation parameter, however. Hence we believe that the bound is correct for higher values of the Coriolis number as we shall argue in §§ 3, 4 and 5 of this paper.

In the following we start with the formulation of the mathematical problem for the case of plane Couette flow in a rotating system. Such a situation can be realized experimentally, for instance, in the narrow fluid gap between two cylinders rotating at nearly the same rate about their common axis when the inner cylinder is rotating slightly faster. Even the case when the cylinders are in addition moving relative to each other in the axial direction is included in our formulation. After energy stability and linear stability have been reviewed in § 3, we turn to the derivation of the upper bound for the momentum transport in § 4. A discussion of the results and their implications will be given in § 5.

2. Mathematical formulation of the problem

We consider simple shear flows in a fluid system rotating with the constant angular velocity $\boldsymbol{\Omega}_D$ about an axis fixed in space. The fluid is regarded as incompressible with a constant kinematic viscosity ν. It is bounded by two infinitely extended rigid walls which are parallel to each other as well as to $\boldsymbol{\Omega}_D$ and which are moving with the constant velocity $\boldsymbol{U}_D$ relative to each other as indicated in figure 1. Using the distance d between the walls as length scale and d^2/ν as time scale we can write the Navier–Stokes equation for the velocity vector $\hat{\boldsymbol{v}}$ in the form

$$\partial_t \hat{\boldsymbol{v}} + \hat{\boldsymbol{v}} \cdot \nabla \hat{\boldsymbol{v}} + \tau \boldsymbol{j} \times \hat{\boldsymbol{v}} + \hat{\tau} \boldsymbol{i} \times \hat{\boldsymbol{v}} = -\nabla \pi + \nabla^2 \hat{\boldsymbol{v}}, \qquad (2.1a)$$

$$\nabla \cdot \hat{\boldsymbol{v}} = 0, \qquad (2.1b)$$

where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ form a right-handed system of orthogonal unit vectors with $\boldsymbol{i}$ in the direction of $\boldsymbol{U}_D$ and $\boldsymbol{k}$ normal to the walls. Accordingly $\hat{\boldsymbol{v}}$ must satisfy the boundary condition

$$\hat{\boldsymbol{v}} = \mp \boldsymbol{i} Re/2 \quad \text{at} \quad z = \pm\tfrac{1}{2}. \tag{2.2}$$

Here the Reynolds number Re and the two Coriolis numbers, τ and $\hat{\tau}$, have been introduced,

$$Re = \frac{|\boldsymbol{U}_D|d}{\nu}, \qquad \tau = \frac{2\boldsymbol{\Omega}_D \cdot \boldsymbol{j} d^2}{\nu}, \quad \hat{\tau} = \frac{2\boldsymbol{\Omega}_D \cdot \boldsymbol{i} d^2}{\nu}. \tag{2.3}$$

Throughout this paper $\tau > 0$ will be assumed while the sign of $\hat{\tau}$ may be arbitrary. A system of coordinates x, y, z parallel to $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ will be used. By taking the average over the x, y-dependence of equation (2.1a), which will be indicated by an overline, we obtain the following equation for the mean flow, $\boldsymbol{U} \equiv \overline{\hat{\boldsymbol{v}}}$:

$$-\partial_t \boldsymbol{U} + \partial^2_{zz}\boldsymbol{U} = \partial_z \overline{(v_z v_x \boldsymbol{i} + v_z v_y \boldsymbol{j})} \tag{2.4}$$

where $\boldsymbol{v} \equiv \hat{\boldsymbol{v}} - \boldsymbol{U}$ has been introduced. Since $\boldsymbol{U} \cdot \boldsymbol{k} \equiv 0$ the Coriolis force does not enter this equation and is balanced entirely by the z-component of the pressure gradient. We now introduce as our definition of turbulence under stationary conditions that averaged quantities are time independent. This definition allows us to integrate immediately equation (2.4),

$$\partial_z \boldsymbol{U} = \overline{(v_z v_x \boldsymbol{i} + v_z v_y \boldsymbol{j})} - \langle v_z v_x \boldsymbol{i} + v_z v_y \boldsymbol{j} \rangle - Re\, \boldsymbol{i}, \tag{2.5}$$

where the angular brackets indicate the average over the fluid layer and where the constant of integration has been determined such that the boundary conditions (2.2) are satisfied.

It is convenient to introduce the general representation for the solenoidal velocity field $\boldsymbol{v}$ with vanishing x, y-average,

$$\boldsymbol{v} = \nabla \times (\nabla \Phi \times \boldsymbol{k}) + \nabla \Psi \times \boldsymbol{k} \equiv \boldsymbol{\delta}\Phi + \boldsymbol{\eta}\Psi, \tag{2.6}$$

where the conditions $\overline{\Phi} = \overline{\Psi} = 0$ can be imposed without losing generality. By taking the z-components of the (curl)^2 and of the curl of equation (2.1a) two equations for Φ and Ψ are obtained:

$$\nabla^4 \Delta_2 \Phi - (\tau \boldsymbol{j} + \hat{\tau}\boldsymbol{i}) \cdot \nabla \Delta_2 \Psi = \boldsymbol{\delta} \cdot (\boldsymbol{v} \cdot \nabla \boldsymbol{v}) + (\partial_t + \boldsymbol{U} \cdot \nabla)\nabla^2 \Delta_2 \Phi - \boldsymbol{U}'' \cdot \nabla \Delta_2 \Phi, \tag{2.7a}$$

$$\nabla^2 \Delta_2 \Psi + (\tau \boldsymbol{j} + \hat{\tau}\boldsymbol{i}) \cdot \nabla \Delta_2 \Phi = \boldsymbol{\eta} \cdot (\boldsymbol{v} \cdot \nabla \boldsymbol{v}) + (\partial_t + \boldsymbol{U} \cdot \nabla)\Delta_2 \Psi - \boldsymbol{U}' \cdot \boldsymbol{\eta} \Delta_2 \Phi, \tag{2.7b}$$

where Δ_2 denotes the two-dimensional Laplacian, $\Delta_2 = \partial^2_{xx} + \partial^2_{yy}$, and $\boldsymbol{U}'(\boldsymbol{U}'')$ indicates the first (second) derivative of $\boldsymbol{U}$ with respect to z.

After multiplying equations (2.7a) and (2.7b) by Φ and Ψ, respectively, and averaging the results over the fluid layer, as indicated by the angular brackets, we obtain the following energy balances for the poloidal and toroidal components of the velocity field:

$$\langle |\boldsymbol{k} \times \nabla \nabla^2 \Phi|^2 \rangle + \langle \Delta_2 \Phi (\tau \boldsymbol{j} + \hat{\tau}\boldsymbol{i}) \cdot \nabla \Psi \rangle + \langle \boldsymbol{\delta}\Phi \cdot [(\boldsymbol{\delta}\Phi + \boldsymbol{\eta}\Psi) \cdot \nabla] \boldsymbol{\eta}\Psi \rangle = \langle \Delta_2 \Phi\, \boldsymbol{U}' \cdot \nabla \partial_z \Phi \rangle, \tag{2.8a}$$

$$\langle |\boldsymbol{k} \times \nabla \nabla \Psi|^2 \rangle - \langle \Delta_2 \Phi (\tau \boldsymbol{j} + \hat{\tau}\boldsymbol{i}) \cdot \nabla \Psi \rangle - \langle \boldsymbol{\delta}\Phi \cdot [(\boldsymbol{\delta}\Phi + \boldsymbol{\eta}\Psi) \cdot \nabla] \boldsymbol{\eta}\Psi \rangle = -\langle \Psi\, \boldsymbol{U}' \cdot \boldsymbol{\eta} \Delta_2 \Phi \rangle, \tag{2.8b}$$

where the boundary conditions

$$\Phi = \partial_z \Phi = \Psi = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2} \tag{2.9}$$

have been used in the partial integrations. When both expressions (2.8) are added the well-known energy balance for the fluctuating component of the velocity is obtained,

$$\langle|\boldsymbol{k} \times \nabla\nabla^2\Phi|^2\rangle + \langle|\boldsymbol{k} \times \nabla\nabla\Psi|^2\rangle - \langle\Delta_2\Phi\, \boldsymbol{U}' \cdot (\nabla_2\partial_z\Phi + \boldsymbol{\eta}\Psi)\rangle = 0, \tag{2.10}$$

where the abbreviation $\nabla_2 \equiv \nabla - \boldsymbol{k}\,\boldsymbol{k}\cdot\nabla$ has been introduced. After insertion of expression (2.5) the relationship

$$\langle|\nabla_2\nabla^2\Phi|^2\rangle + \langle|\nabla_2\nabla\Psi|^2\rangle + \langle|\overline{w(\nabla_2\partial_z\Phi + \boldsymbol{\eta}\Psi)} - \langle w(\nabla_2\partial_z\Phi + \boldsymbol{\eta}\Psi)\rangle|^2\rangle$$
$$= Re\langle w\left(\partial^2_{xz}\Phi + \partial_y\Psi\right)\rangle \tag{2.11}$$

is obtained where the abbreviation $w = -\Delta_2\Phi$ and the identity $\langle f(f - \langle f\rangle)\rangle = \langle|f - \langle f\rangle|^2\rangle$ for any function $f(z)$ have been used.

3. Results of energy and linear stability analysis

Before starting the derivation of upper bounds for the turbulent momentum transport based on equation (2.11) we should recall the analysis leading to the energy stability Reynolds number Re_E and the critical Reynolds number Re_c for the problem described by (2.1) and (2.2). Since $\boldsymbol{U}_0 = -\boldsymbol{i}Rez$ is the basic solution of the problem we obtain for any perturbation $\check{\boldsymbol{v}} \equiv \hat{\boldsymbol{v}} - \boldsymbol{U}_0$ the equations

$$\partial_t\check{\boldsymbol{v}} + (\check{\boldsymbol{v}} - \boldsymbol{i}Rez)\cdot\nabla\check{\boldsymbol{v}} - \check{\boldsymbol{v}}\cdot\boldsymbol{k}\boldsymbol{i}Re + \tau\boldsymbol{j} \times \check{\boldsymbol{v}} + \hat{\tau}\boldsymbol{i} \times \check{\boldsymbol{v}} = -\nabla\check{\pi} + \nabla^2\check{\boldsymbol{v}}, \tag{3.1a}$$
$$\nabla\cdot\check{\boldsymbol{v}} = 0. \tag{3.1b}$$

After multiplication of (3.1*a*) by $\check{\boldsymbol{v}}$ and averaging it over the fluid layer the Reynolds–Orr energy equation,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle|\check{\boldsymbol{v}}|^2\rangle = -\langle|\nabla\check{\boldsymbol{v}}|^2\rangle + Re\langle\check{v}_z\check{v}_x\rangle, \tag{3.2}$$

is obtained. The energy Reynolds number Re_E, which guaranties that for $Re < Re_E$ any finite-amplitude disturbance $\check{\boldsymbol{v}}$ decays exponentially, is determined as the minimum of the variational functional

$$\mathscr{R}_{\mathscr{E}}(\boldsymbol{u}) \equiv \frac{\langle|\nabla\boldsymbol{u}|^2\rangle - 2\langle\check{\pi}\nabla\cdot\boldsymbol{u}\rangle}{\langle u_x u_z\rangle} \tag{3.3}$$

among all vector fields $\boldsymbol{u}$ that satisfy the boundary condition $\boldsymbol{u} = 0$ at $z = 1/2$ and $\langle u_x u_z\rangle > 0$; $\check{\pi}$ is the Lagrange multiplying function which takes into account the constraint $\nabla\cdot\boldsymbol{u} = 0$. The Euler–Lagrange equations for a stationary value P of the functional (3.3) are given by

$$-(\boldsymbol{i}u_z + \boldsymbol{k}u_x)P/2 = -\nabla\check{\pi} + \nabla^2\boldsymbol{u}, \tag{3.4a}$$
$$\nabla\cdot\boldsymbol{u} = 0. \tag{3.4b}$$

Anticipating that the solution of these equations that minimizes P depends only on y and z we take the x-component of the curl of (3.4*a*),

$$\theta P/2 = \nabla^4\psi, \tag{3.5}$$

where we have introduced $u_y = \partial_z\psi(y, z), u_z = -\partial_y\psi(y, z)$ and $\theta(y, z) = \partial_y u_x$. The y-derivative of the x-component of (3.4*a*) yields

$$\partial^2_{yy}\psi P/2 = \nabla^2\theta. \tag{3.6}$$

The solutions of (3.5) and (3.6) together with the boundary conditions

$$\psi = \partial_z \psi = \theta = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2} \tag{3.7}$$

are identical to those for the onset of convection in a Rayleigh–Bénard layer. We thus obtain as minimum value of P

$$P^2/4 = 1708 \quad \text{or} \quad Re_E = 2\sqrt{1708}. \tag{3.8}$$

Here 1708 denotes the well-known value of the critical Rayleigh number (Chandrasekhar 1961) of convection in the presence of no-slip boundaries. For the proof that x-independent solutions of equations (3.4) do indeed provide the minimum value of P see Busse (1972).

It turns out that the linear analysis of the stability of the basic solution $\boldsymbol{U}_0 = -\boldsymbol{i}Rez$ with respect to infinitesimal disturbances $\tilde{\boldsymbol{v}}$ yields a critical value Re_c of the Reynolds number which coincides with Re_E for the special rotation rate $\tau = \sqrt{1708}$, which can be easily seen when the stability equations

$$\partial_t \tilde{\boldsymbol{v}} - \boldsymbol{i}Rez \cdot \nabla \tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}} \cdot \boldsymbol{k}\,\boldsymbol{i}Re + \tau \boldsymbol{j} \times \tilde{\boldsymbol{v}} + \hat{\tau} \boldsymbol{i} \times \tilde{\boldsymbol{v}} = -\nabla \tilde{\pi} + \nabla^2 \tilde{\boldsymbol{v}}, \tag{3.9a}$$

$$\nabla \cdot \tilde{\boldsymbol{v}} = 0, \tag{3.9b}$$

are considered. Anticipating that the lowest value of Re corresponding to non-decaying solutions of these equations is given by x-independent steady solutions we take the x-component of the curl of (3.9a) and the y-derivative of the x-component of (3.9a),

$$\tau \tilde{\theta} = \nabla^4 \tilde{\psi}, \tag{3.10a}$$

$$(Re - \tau)\partial^2_{yy} \tilde{\psi} = \nabla^2 \tilde{\theta}, \tag{3.10b}$$

where $\tilde{v}_y = \partial_z \tilde{\psi}(y, z)$, $\tilde{v}_z = -\partial_y \tilde{\psi}(y, z)$ and $\tilde{\theta}(y, z) = \partial_y \tilde{v}_x$ have been used. Again, the solutions of (3.10) together with the boundary conditions

$$\tilde{\psi} = \partial_z \tilde{\psi} = \tilde{\theta} = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2} \tag{3.11}$$

are identical to those for the onset of convection in a Rayleigh–Bénard layer. We thus obtain as minimum value for $\tau(Re - \tau)$

$$\tau(Re - \tau) = 1708, \quad \text{yielding} \quad Re_c = 1708/\tau + \tau. \tag{3.12}$$

As a function of τ, Re_c reaches its minimum value for $\tau_E = \sqrt{1708}$ at which point it coincides with Re_E according to relationship (3.8). For this reason the x-independent steady disturbances do indeed correspond to Re_c, at least at $\tau = \tau_E$, as we had anticipated. Experimental observations also indicate that relationship (3.12) is generally valid. For small values of τ, as Re_c tends to infinity, finite-amplitude x-dependent disturbances are observed, at least in the case $\hat{\tau} = 0$, just as in non-rotating plane Couette flow. Neither theoretical results nor experimental evidence (for a recent review of theoretical and experimental results for the Taylor–Couette problem refer to Dubrulle *et al.* 2005) however, seem to contradict the validity of the relationship

$$Re_E = 1708/\tau + \tau \quad \text{for} \quad \tau \gtrsim \sqrt{1708}. \tag{3.13}$$

A rigorous mathematical proof of this relationship for values other than $\tau = \tau_E$ is not available, unfortunately. Kaiser & von Wahl (1996) have shown that x-independent disturbances of arbitrary amplitude must decay for $Re < 1708/\tau + \tau$, but this is far from establishing hypothesis (3.13). As has already been shown by Busse (1970b) the results of this section hold for arbitrary values of $\hat{\tau}$.

4. Upper bound on the momentum transport in turbulent Couette flow

The momentum transport across the fluid layer in the x-direction is given by $M = Re + \langle v_z v_x \rangle$ according to equation (2.5). Since according to (2.11), $\langle v_z v_x \rangle \geqslant 0$, an upper bound on this quantity is of primary interest. Instead of looking for an upper bound μ on the quantity $\langle v_z v_x \rangle$ at a given value Re of the Reynolds number, however, we shall look for a lower bound R on Re at a given value μ of the turbulent contribution to the momentum transport, since both quantities are monotonically related as can be verified from the results. As in earlier work (Busse 1970*a*) we are thus led to the formulation of the variational problem:

Find the minimum $R(\mu, \tau)$ *of the functional*

$$\mathscr{R}(\Phi, \Psi, \mu, \tau) \equiv \frac{\langle |\nabla_2 \nabla^2 \Phi|^2 \rangle + \langle |\nabla_2 \nabla \Psi|^2 \rangle}{\langle w\left(\partial^2_{xz}\Phi + \partial_y \Psi\right)\rangle} + \mu \frac{\langle |\overline{w(\nabla_2 \partial_z \Phi + \boldsymbol{\eta}\Psi)} - \langle w(\nabla_2 \partial_z \Phi + \boldsymbol{\eta}\Psi)\rangle|^2 \rangle}{\langle w\left(\partial^2_{xz}\Phi + \partial_y \Psi\right)\rangle^2} \tag{4.1}$$

among all fields $\Phi (w \equiv -\Delta_2 \Phi)$ *and* Ψ *that satisfy the boundary conditions* (2.9), *condition* (2.8*a*) *and the condition* $\langle w(\partial^2_{xz}\Phi + \partial_y \Psi)\rangle > 0$.

In considering this variational problem we anticipate that the term $\langle |\overline{w(\partial^2_{yz}\Phi - \partial_x \Psi)} - \langle w(\partial^2_{yz}\Phi - \partial_x \Psi)\rangle|^2 \rangle$ vanishes for the minimizing field since it contributes only a positive term in the functional (4.1). It thus will be dropped in the following. Adopting the ideas expressed in the preceding section we further introduce the hypothesis that for moderate and large positive values of τ the minimizing fields Φ and Ψ do not depend on the coordinate x. In this case the functional and the side constraint (2.8*a*) simplify significantly. We use the new variable $\Theta \equiv \partial_y \Psi$ and introduce the side constraint (2.8*a*) with the Lagrange multiplier λ into the functional, which thus assumes the form

$$\mathscr{R}(\Phi, \Psi, \mu, \tau) \equiv \frac{\langle |\nabla_2 \nabla^2 \Phi|^2 \rangle (1+\lambda) + \langle |\nabla \Theta|^2 \rangle}{\langle w\Theta \rangle} - \lambda \tau + \mu \frac{\langle |\overline{w\Theta} - \langle w\Theta \rangle|^2 \rangle}{\langle w\Theta \rangle^2}. \tag{4.2}$$

Since, except for the positive numerator of the first term on the right-hand side of (4.2), the functional does not change when Φ and Θ are multiplied by δ and δ^{-1}, respectively, the numerator may be minimized with respect to the arbitrary factor δ with the result

$$\mathscr{R}(\Phi, \Psi, \mu, \tau) \equiv \frac{2\sqrt{\langle |\nabla_2 \nabla^2 \Phi|^2 \rangle (1+\lambda) \langle |\nabla \Theta|^2 \rangle} - \lambda \tau \langle w\Theta \rangle}{\langle w\Theta \rangle} + \mu \frac{\langle |\overline{w\Theta} - \langle w\Theta \rangle|^2 \rangle}{\langle w\Theta \rangle^2}. \tag{4.3}$$

The best bound is obtained when the minimum R as a function of the parameter λ reaches a maximum. This motivates us to search for the extremum of the functional (4.3) as a function of λ. This procedure yields the maximizing value of λ

$$\lambda = \frac{\langle |\nabla_2 \nabla^2 \Phi|^2 \rangle \langle |\nabla \Theta|^2 \rangle}{\tau^2 \langle w\Theta \rangle^2} - 1. \tag{4.4}$$

The introduction of this expression into the functional (4.3) yields

$$\mathscr{R}(\Phi, \Psi, \mu, \tau) \equiv \frac{\langle |\nabla_2 \nabla^2 \Phi|^2 \rangle \langle |\nabla \Theta|^2 \rangle}{\tau \langle w\Theta \rangle^2} + \tau + \mu \frac{\langle |\overline{w\Theta} - \langle w\Theta \rangle|^2 \rangle}{\langle w\Theta \rangle^2}. \tag{4.5}$$

Since τ is a given positive parameter we shall consider instead of the variational functional (4.1) the closely related functional

$$\hat{\mathscr{R}}(\Phi, \Theta, \hat{\mu}) \equiv (\mathscr{R}(\Phi, \Psi, \mu, \tau) - \tau)\tau \equiv \frac{\langle|\nabla_2\nabla^2\Phi|^2\rangle\langle|\nabla\Theta|^2\rangle}{\langle w\Theta\rangle^2} + \hat{\mu}\frac{\langle|\overline{w\Theta} - \langle w\Theta\rangle|^2\rangle}{\langle w\Theta\rangle^2}, \quad (4.6)$$

where the definition $\hat{\mu} \equiv \tau\mu$ has been introduced. In the following we take advantage of the fact that the functional $\hat{\mathscr{R}}$ no longer depends on τ explicitly. In fact this functional is identical to the one the minimum of which provides a lower bound for the Rayleigh number at a given value $\hat{\mu}$ of the convective heat transport in a fluid layer heated from below (Howard 1963; Busse 1969). The Euler–Lagrange equations for a stationary value of the functional (4.6) cannot be solved exactly, but approximate solutions based on hierarchies of N boundary layers have been described by Busse (1969). Solutions of the form

$$\Phi = \Phi^{(N)} \equiv \sum_{n=1}^{N} \phi_n(y) w_n(z)/\alpha_n^2, \quad \Theta = \Theta^{(N)} \equiv \sum_{n=1}^{N} \phi_n(y)\theta_n(z) \quad (4.7)$$

have been obtained with the functions $\phi_n(y)$ satisfying

$$\partial^2_{yy}\phi_n(y) = -\alpha_n^2\phi_n(y), \quad \langle|\phi_n(y)|^2\rangle = 1. \quad (4.8)$$

For these multi-α-solutions asymptotic expressions for the minimum of the functional (4.6),

$$\hat{R}^{(N)}(\hat{\mu}) = (3 \times 4^N - 1)(4^N - 1)b_1^4\hat{\mu}^{2/(3-4^{-N})} \quad (4.9)$$

have been derived where b_1 is given by

$$b_1^{4(3-4^{-N})} = 4^{-6N}(\sigma/\beta)^3\left(\beta 4^{3/4}\right)^{4(1-4^{-N})}(1-4^{-N})^{-2}. \quad (4.10)$$

Here σ and β are constants of the order unity which are given in Busse (1969). The absolute minimum $\hat{Re}$ of the functional $\hat{\mathscr{R}}$ defined by

$$\hat{R}(\hat{\mu}) = \min_N \hat{R}^{(N)}(\hat{\mu}) \quad (4.11)$$

is given with increasing $\hat{\mu}$ by $\hat{R}^{(N)}(\hat{\mu})$ with N being all positive integers starting with $N = 1$.

5. Discussion

Although a firm proof is lacking for the hypothesis that x-dependent trial fields Φ, Ψ do not lead to lower values of the functional (4.1) at a given value of μ for $\tau \gtrsim \sqrt{1708}$, we believe that the result

$$Re \geqslant R(\mu, \tau) \equiv \min\{\mathscr{R}(\Phi, \Psi, \mu, \tau)\} \equiv \tau + \min\{\hat{\mathscr{R}}(\Phi, \Theta, \mu\tau)\}/\tau \quad (5.1)$$

for any deviation from the basic solution $\boldsymbol{U}_0 = -\boldsymbol{i}Rez$ of rotating plane Couette flow is correct for $\tau \gtrsim \sqrt{1708}$. Since the minimum of $\hat{\mathscr{R}}(\Phi, \Theta, \mu\tau)$ for $\mu = 0$ is given by 1708, the result (5.1) coincides with the relationship (3.12) for the critical value of the Reynolds number in the limit $\mu \rightarrow 0$. Relationship (5.1) is more general, of course, since it provides the upper bound for the momentum transport for values $Re > \tau + 1708/\tau$, which is implied by the results (4.9), (4.11). In the limit of large values of Re, $Re \gg \tau \gtrsim \sqrt{1708}$, the bound on the momentum transport assumes the form (see, for instance, equation (4.2) of Busse 1969)

$$\mu \equiv \langle v_z v_x \rangle \leqslant \big((Re-\tau)\tau^{1/3}/10.114\big)^{3/2}. \tag{5.2}$$

It is noteworthy that a similar upper bound for the momentum transport can be obtained if the functional

$$\tilde{\mathscr{R}}(\Phi, \Theta, \hat{\mu}) \equiv \frac{\langle |\mathrm{d}\Delta_2 \Phi/\mathrm{d}z|^2\rangle \langle |\mathrm{d}\Theta/\mathrm{d}z|^2\rangle}{\langle w\Theta\rangle^2} + \hat{\mu}\frac{\langle |\overline{w\Theta} - \langle w\Theta\rangle|^2\rangle}{\langle w\Theta\rangle^2}. \tag{5.3}$$

is considered instead of $\hat{\mathscr{R}}(\Phi, \Theta, \mu\tau)$, as has been done by Howard (1963). The functional (5.3) arises in the theory of upper bounds on the convective heat transport in the case when the equation of continuity is dropped as a constraint on the trial fields. Evidently $\tilde{\mathscr{R}}(\Phi, \Theta, \hat{\mu}) < \hat{\mathscr{R}}(\Phi, \Theta, \mu\tau)$ holds and the Euler–Lagrange equations for the purely z-dependent trial fields Φ, Θ can be solved analytically in terms of complete elliptic integrals as demonstrated by Howard (1963). In the present context the asymptotic bound for the momentum transport $\mu = \hat{\mu}/\tau$ is of interest and is given by

$$\mu \equiv \langle v_z v_x \rangle < \big((Re-\tau)(3\tau)^{1/3}/4\big)^{3/2}. \tag{5.4}$$

While this upper bound exceeds the upper bound (5.2) as it must, the power-law dependence on $(Re-\tau)\tau^{1/3}$ remains the same.

One of the most fascinating aspects of the theory of upper bounds is the similarity between properties of the experimentally observed turbulence and of the extremalizing vector fields (Busse 1969, 1970*a*, 2002). Here we restrict attention to the mean flow profile that results from the momentum transport of the extremalizing vector fields. Ignoring the boundary layers we find that the mean interior shear is described by

$$\begin{aligned} \frac{\mathrm{d}\hat{U}_x}{\mathrm{d}z} &= \overline{w\Theta} - \langle w\Theta\rangle - R \\ &= \hat{\mu}(\tilde{w}_1\tilde{\theta}_1 - 1)/\tau - (\hat{R} + \tau^2)/\tau \\ &= \big(\hat{R}^{(N)}(\hat{\mu}) - (4^N - 1)b_1^4\hat{\mu}^{2/(3-4^{-N})}\big)/\tau - \big(\hat{R}^{(N)}(\hat{\mu}) + \tau^2\big)/\tau \\ &= -\tau - \hat{R}^{(N)}(\hat{\mu})/(3\times 4^N - 1) \end{aligned} \tag{5.5}$$

where we have used expressions (3.30) and (3.33) of Busse (1969). Since the last term in these equations becomes negligible in the limit of large $\hat{\mu}$ we find the result that $\mathrm{d}\hat{U}_x/\mathrm{d}z$ approaches $-\tau$ for large $\mu\tau$, i.e. the mean flow profile becomes that of a curl-free potential vortex. The realization of linear mean flow profiles with nearly vanishing absolute vorticity is a well-known phenomenon of experiments on turbulent shear flows in rotating systems as demonstrated, for example, by Johnston, Halleen & Lezius (1972). The analogy between the curl-free mean flow profile and that of the isothermal mean temperature in a turbulent convection layer on which relationship (5.5) is based has also been emphasized in the discussion of the numerical simulations of Tanaka *et al.* (2000).

In conclusion we state as the main results of the present paper that considerable evidence exists that for $\tau \gtrsim \sqrt{1708}$ no stationary turbulent states of motion exist within the regime $-Re_E \leqslant Re \leqslant 1708/\tau + \tau$ and that for values of Re exceeding the right-hand side of this inequality the momentum transport is bounded by relationship (5.2).

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